REFLECTION OF SURFACE WAVES BY SUBMERGED CYLINDERS

by

John Grue and Enok Palm
Department of Mechanics, University of Oslo,
Blindern, Oslo 3, Norway

ABSTRACT

Reflection from submerged cylinders are studied by means of integral equations. By expressing the solution as a distribution of vortices, the integral equations become non-singular for closed contours. It is shown that the method gives a short and easy proof for the classical result that no reflection occurs for the circular cylinder. The reflection power for the elliptic contour and the flat plate are studied when the bodies are situated deeply below the surface.

1. Introduction

We consider a fluid bounded by a horizontal free surface. The fluid is of infinite depth and of infinite lateral extent. The problem is assumed to be two-dimensional, taking place in a x-y plane. The x-axis is chosen along the undisturbed free surface and the y-axis positive upwards. The fluid is supposed to be incompressible and the motion irrotational. The velocity $\vec{v}$ may then be written

$$\vec{v} = \nabla\phi \tag{1.1}$$

where ϕ is the velocity potential. ϕ satisfies the two-dimensional Laplacian

$$\nabla^2\phi = 0 \tag{1.2}$$

All the equations will be linearized. The boundary condition of the free surface is then

$$\phi_{tt} + g\phi_y = 0 \qquad y=0 \tag{1.3}$$

ϕ also fulfills the boundary condition

$$\lim_{y\to-\infty} |\nabla\phi| = 0 \tag{1.4}$$

We shall investigate the reflection power of various rigid and non-moving submerged bodies. The boundary condition at the body is then

$$\frac{\partial\phi}{\partial n} = 0 \tag{1.5}$$

where $\partial/\partial n$ denotes the normal derivative. Let us consider the motion due to an incoming sinusoidal wave at $x\to-\infty$ of given frequency σ. ϕ is then proportional to $\exp(i\sigma t)$ where t denotes time. (1.3) reduces to

$$\phi_y - \nu\phi = 0 \qquad y=0 \tag{1.6}$$

where

$$\nu = \sigma^2/g \tag{1.7}$$

Due to the submerged body some of the energy flux is reflected whereas the rest of the energy flux is transmitted as an outgoing wave at $x\to\infty$. The surface elevation due to this last wave, will be of the form

$$A\cos(\nu x-\sigma t+\alpha) \tag{1.8}$$

where A is the amplitude and α a phase constant.

To study the problem, we shall apply the method of integral equations. The standard approach is to use Green's theorem and end up with a singular integral equation. We prefer here to apply the method developed by Kochin (1937) and Keldysh and Lavrentev (1937) (see Wehausen and Laitone, 1960, p.534-536). In their method the solution is expressed as a distribution of vortices over the contour B of the body. This approach has the merit that for closed contours we end up with a non-singular integral equation of Fredholm type of second kind. For analytical treatment this is of considerable advantage. For an open contour, however, the method leads to a singular integral equation of the first kind. It may be worth to mention that a non-singular integral equation also may be obtained by considering a distribution of dipoles.

It simplifies the mathematics to assume (for the moment) that at $x\to\infty$ the motion is a standing wave, instead of an outgoing wave. We then have, instead of (1.8),

$$\phi = \frac{Ag}{\sigma} e^{\nu y}\cos(\nu x+\alpha)\cos \sigma t \qquad x \to +\infty \tag{1.9}$$

Introducing the complex potential $f(z)$ where

$$z = x + iy \qquad \text{Re } f(z) = \phi(x,y) \tag{1.10}$$

with Re indicating the real part, the boundary conditions (1.3)-(1.5) may be written

$$\mathrm{Im}\{f'(x)+i\nu f(x)\} = 0 \tag{1.11}$$

$$\mathrm{Im}\{f'(\zeta(s))e^{i\beta(s)}\} = 0 \tag{1.12}$$

$$\lim_{y\to-\infty} |f'| = 0 \tag{1.13}$$

Here the contour of the submerged body is given parametrically by $z=\zeta(s)$ with s denoting the arc length. $\beta(s)$ is defined by

$$e^{i\beta(s)} = \frac{d\zeta}{ds} \tag{1.14}$$

and is the angle between the tangent vector and the positive x-direction (see Fig. 1). Writing $f(z)$ in the form

$$f(z) = f_1(z) + \frac{Ag}{\sigma}\, e^{-i(\nu z+\alpha)} \tag{1.15}$$

$f_1(z)$ must satisfy

$$\lim_{x\to+\infty} f_1(z) = 0 \tag{1.16}$$

$$\mathrm{Im}\{[f_1'(\zeta(s)) - i\,\frac{Ag\nu}{\sigma}\, e^{-i\nu\zeta(s)}]e^{i\beta(s)}\} = 0 \tag{1.17}$$

as well as the free surface condition and (1.13).

The complex velocity potential for a vortex of strength Γ located at $z=z_0$ and fulfilling the boundary conditions (1.13), the free surface condition and (1.16), is given by

$$f_\nu(z,z_0) = \frac{\Gamma}{2\pi i}\{\log(z-z_0)(z-\bar{z}_0)-2e^{-i\nu z}\int_{+\infty}^{z} \frac{e^{i\nu u}}{u-\bar{z}_0}\,du\} \tag{1.18}$$

where a bar denotes complex conjugate. We now write $f_1(z)$ in the form

$$f_1(z) = \int_B \gamma(s)f_\nu(z,\zeta(s))ds \tag{1.19}$$

$\gamma(s)$ is real and is the unknown strength of the vortex distribution. It turns out that for a closed contour $\gamma(s)$ is the (unknown) tangential velocity. (1.19) satisfies all the conditions except the boundary condition (1.17) at the contour. It is easily shown that also this condition is fulfilled if γ satisfies the integral equation

$$-\gamma(s') + 2\fint_B \gamma(s) f'_\nu(\zeta(s'),\zeta(s)) e^{i\beta(s')} ds =$$
$$2iA\sigma e^{-i(\nu\zeta(s')-\beta(s')+\alpha)} \qquad (1.20)$$

where a bar at the integral sign indicates the principal value. Since $\gamma(s)$ is real, (1.20) gives two different integral equations for determining $\gamma(s)$. The imaginary part gives a singular equation of the first kind

$$\frac{1}{\pi}\fint_B \gamma(s)K(s',s)ds = -\mathrm{Im}(2iA\sigma e^{-i(\nu\zeta(s')-\beta(s')+\alpha)}) \qquad (1.21)$$

where Im denotes the imaginary part.

The real part gives a (non-singular) Fredholm equation of the second kind

$$-\gamma(s') + \frac{1}{\pi}\int_B \gamma(s)L(s',s)ds = \mathrm{Re}(2iA\sigma e^{-i(\nu\zeta(s')-\beta(s')+\alpha)}) \qquad (1.22)$$

Here K and L are defined by

$$K = \mathrm{Re}\ \chi, \quad L = \mathrm{Im}\ \chi \qquad (1.23)$$

where

$$\chi = \frac{e^{i\beta(s')}}{\zeta(s')-\zeta(s)} - \frac{e^{i\beta(s')}}{\zeta(s')-\overline{\zeta(s)}} +$$
$$2i\nu e^{i\beta(s')-i\nu\zeta(s')} \int_\infty^{\zeta(s')} \frac{e^{i\nu u}}{u-\overline{\zeta(s)}} du \qquad (1.24)$$

The equations (1.20)-(1.22) are derived in the article by Wehausen and Laitone (1960). (In (1.22) we have corrected the right hand side for an obvious misprint). It is rather easy to demonstrate that (1.22) is non-singular. A possible singularity is due to the first term in (1.24). The integral of this term may, however, be written

$$\operatorname{Im} \int_B \frac{\gamma(s)e^{i\beta(s')}}{\zeta(s')-\zeta(s)}\, ds = \int_B \frac{\gamma(s)\cos(n',r)}{r}\, ds \tag{1.25}$$

where r is the radius vector from $\zeta(s)$ to $\zeta(s')$, and (n',r) is the angle between the radius vector and the outward normal at the point $\zeta(s')$.

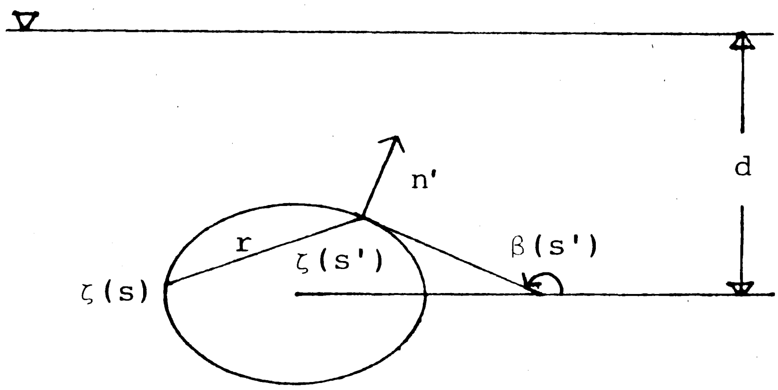

Figure 1. Submerged body under a free surface

The right hand side in (1.25) is non-singular since $\cos(n',r) \to 0$ when $r \to 0$. Indeed, the kernel $\cos(n',r)/r$ is a continuous function.

It is readily shown from (1.21), or (1.22), that

$$\int_B \gamma(s)ds = 0 \tag{1.26}$$

for a closed contour. Physically this equation expresses the fact that the velocity circulation for the contour is zero. This is a result of the problem being a Neumann problem. To solve (1.22) we may apply the theory of Fourier transform. It is then appropriate to introduce an angle variable instead of s.

We shall assume that a unique solution of (1.22) exists. Since (1.22) is a Fredholm equation of second kind, a unique solution exists if the corresponding homogeneous equation has only the trivial solution $\gamma=0$. This seems obvious to be true from physical arguments. A strict existence proof may be given when the body is located sufficiently deep below the free surface and the wave number ν is not too small.

2. The circular cylinder

We shall in this and the next two sections apply the formulas in section 1 to three different contours. First we assume that the contour has the form of a circle (circular cylinder). This problem is a classical one and has been solved in papers by Dean (1948), Ursell (1950) and Ogilvie (1963). We also mention a recent paper by Mehlum (1980) where the solution is obtained in a closed form. It is found in these papers that there is no reflection from the cylinder, independent of the values of the frequency σ and the location of the cylinder.

We shall now show that this result may be obtained from the formulas in section 1 with a minimum of mathematical effort. First we consider (1.19) for $x \to -\infty$. Applying (1.26) we notice that the logarithmic terms give contributions of $O(x^{-1})$. The important term is the third one in (1.18) which by contour integration is found to be

$$2\Gamma e^{-i\nu z+i\nu\bar{z}_0} \tag{2.1}$$

Introducing this in (1.19), we have

$$f_1(z) = 2e^{-i\nu z}\int_B \gamma(s)e^{i\nu\bar{\zeta}(s)}ds \qquad x \to -\infty \tag{2.2}$$

Pure formally we may write

$$f_1(z) = Pe^{-i\nu z - i\alpha + i\delta} \qquad x \to -\infty \tag{2.3}$$

where P and δ are real constants. Returning now to (1.9), we replace α by $\alpha - \frac{\pi}{2}$ and $\cos \sigma t$ by $\sin \sigma t$ and add the expressions. We thereby obtain an outgoing wave (with amplitude Ag/σ for the velocity potential) at $x \to \infty$.

Let us assume that P and δ are independent of α. Taking the real part of (2.3) multiplied with $\cos \sigma t$, adding the expression obtained by replacing α by $\alpha - \frac{\pi}{2}$ and multiplying with $\sin \sigma t$, we obtain for the velocity potential an incoming wave of the form

$$Pe^{\nu y}\cos(\nu x - \sigma t + \alpha - \delta) \qquad x \to -\infty \tag{2.4}$$

and no reflected wave. Noting that the energy flux of the incoming wave at $x \to -\infty$ must be equal to the energy flux of the outgoing wave at $x \to +\infty$, P must be equal to Ag/σ. In (2.4) δ is the phase difference between the incoming and outgoing wave. To prove that no reflection occur, it is sufficient to prove that P and δ are independent of α.

Let d denote the distance between the free surface and the center of the circle. The contour may then be parameterized by

$$\zeta(\theta) = ae^{i\theta} - id \tag{2.5}$$

where a denotes the radius of the circle. γ is now considered as a function of θ and due to (1.26) we may write

$$\gamma(\theta) = \sum_{m=1}^{\infty} (C_m e^{im\theta} + \bar{C}_m e^{-im\theta}) \tag{2.6}$$

If now (2.5) and (2.6) are introduced in (2.2) and the integration is carried out, we note that only C_m appears in the formula for $f_1(z)$ when $x \to -\infty$. We shall show that C_m is proportional to $\exp(-i\alpha)$, which involves that P and δ are independent of α.

Returning to (1.22) and replacing s by θ, we multiply the equation with $\exp(-in\theta')$ where n is a positive integer. By integration from 0 to 2π, we note that the right hand side is proportional to $\exp(-i\alpha)$. The term $-\gamma(s')$ gives rise to a term $-\pi C_n$. The integral on the left hand side of the integral equation is divided into three parts. The first part is given by (1.25). It is immediately seen that for a circle $\cos(n',r)/r$ is constant and equal to $1/2a$. From (1.26) it follows that this term gives no contribution. The second term may by (2.5) be written

$$\frac{a}{\pi}\operatorname{Im}\int_0^{2\pi}\gamma(\theta)\,\frac{e^{i\beta(\theta')}}{2id(1+\frac{ia}{2d}(e^{i\theta'}-e^{-i\theta}))}\,d\theta \tag{2.7}$$

Since $a/2d<1$ the integrand may be expanded in a covergent power series with respect to $ia(\exp(i\theta')-\exp(-i\theta))/2d$. It follows that (2.7) contains terms of the type $C_m\exp(ip\theta')$ and $\bar{C}_m\exp(-ip\theta')$ when it is integrated, p is a positive integer. When (2.7) is multiplied with $\exp(-in\theta')$ and integrated from 0 to 2π, we conclude that only the part of γ corresponding to C_m-terms gives contributions different from zero.

The last integral may be written

$$\frac{2a\nu}{\pi}\operatorname{Im}\{ie^{i\beta(\theta')-i\nu\zeta(\theta')}\int_0^{2\pi}\gamma(\theta)d\theta\int_{\infty}^{\zeta(\theta')}\frac{e^{i\nu u}du}{u-\overline{\zeta(\theta)}}\} \tag{2.8}$$

Changing the order of integration, the inner integral takes the form

$$\int_0^{2\pi}\frac{\gamma(\theta)d\theta}{u-id-ae^{-i\theta}} \tag{2.9}$$

Since the path of integration in the complex z-plane from $+\infty$ to $\zeta(\theta')$ always makes $|u-id|>a$, the integrand may as in (2.7) be expanded in a convergent power series with respect to $a\exp(-i\theta)/(u-id)$. Thereby we see that the expression (2.8) contains terms of the type $C_m\exp(ip\theta')$ and $\bar{C}_m\exp(-ip\theta)$ (p is a positive integer), and when (2.8) is multiplied by $\exp(-in\theta')$ and integrated from 0 to 2π we get contribution from the part of γ corresponding to the C_m terms only. The integral equation therefore leads to an infinite system of equations of the form

$$-\pi C_n + \sum_{m=1}^{\infty} a_{nm} C_m = T_n e^{-i\alpha} \qquad n=1,2,\ldots$$

with no coupling between C_m and $\bar{C}_m$ and where a_{nm} and T_n are independent of α. Thereby C_m is proportional to $\exp(-i\alpha)$ and we have shown that no reflection occurs.

To examine the efficiency of the method, we calculated the phase difference δ for νa varying from 0 to 2π, see Fig. 2. For the worst case, $\nu a=2\pi$, we had to apply 20 terms to obtain an error less than about 1 pr.mille. The results are very close to those found by Mehlum (1980).

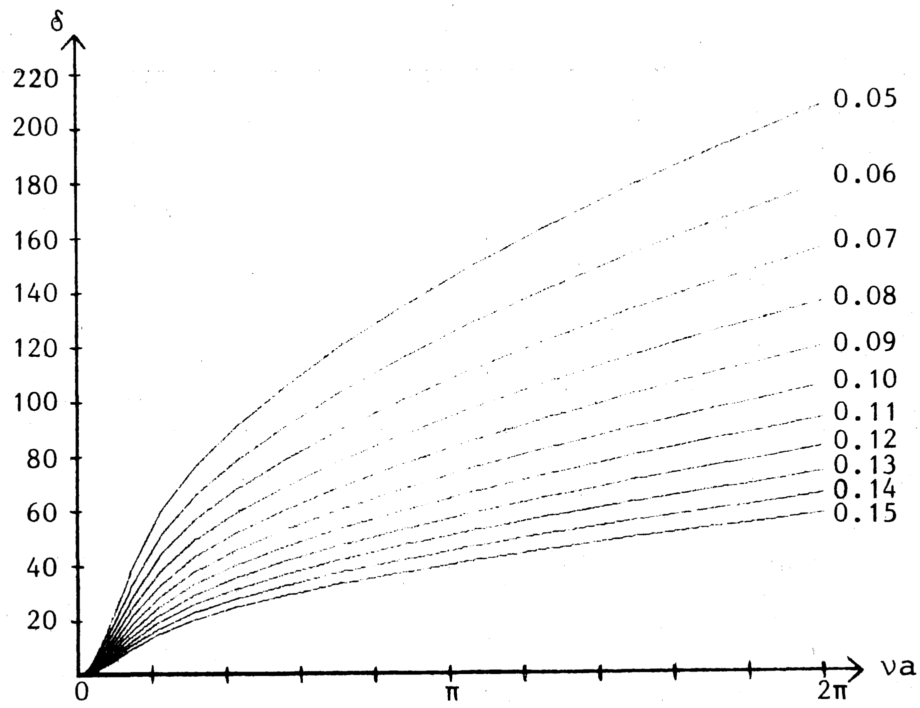

Figure 2. Phase difference δ in degrees. Curves plotted for $(\nu d-\nu a)/2\pi$=const.=0.05,0.06,...,0.15 and νa varying from 0 to 2π.

3. The flat plate

We consider here the reflection from a flat plate parallel to the free surface. It is easily shown that for contours which are not closed, the two integral equations (2.21) and (2.22) reduces to one, viz. (2.21). It is found appropriate to introduce non-dimensional quantities. As characteristic length scale we apply half the length of the plate, a say. The velocity potential is made dimensionless by dividing with Ag/σ (see (1.9)). Let a "hat" indicate quantities with dimension, the dimensionless quantities are then defined by

$$x = \hat{x}/a, \quad y = \hat{y}/a$$
$$d = \hat{d}/a, \quad k = \nu a$$
$$\phi = \hat{\phi}/\frac{Ag}{\sigma}$$

Here $\hat{d}$ is the distance between the free surface and the flat plate.

Considering first the case of a standing wave, the velocity potential at $x \to \infty$ now takes the form

$$\phi = e^{ky}\cos(kx+\alpha)\cos \sigma t \qquad x\to\infty \tag{3.1}$$

Writing $\zeta=\xi-id$, the integral equation (1.21) takes the form

$$\frac{1}{2\pi}\int_{-1}^{1}\gamma(\xi)K(x,\xi)d\xi = -ke^{-kd}\cos(kx+\alpha) \qquad |x|<1 \tag{3.2}$$

where γ now is the difference between the tangential velocity at the lower and upper side. Here

$$K(x,\xi)=\mathrm{Re}\left[\frac{1}{x-\xi} - \frac{1}{x-\xi-2id} + 2ike^{-kd-ikx}\int_{\infty}^{x-id}\frac{e^{iku}}{u-id-\xi}du\right] \tag{3.3}$$

We shall assume that the velocity circulation is zero. Hence

$$\int_{-1}^{1}\gamma(\xi)d\xi = 0 \tag{3.4}$$

Applying (3.4) we note that the three terms in (3.3) are of the order

$$O(1),\ O(d^{-2}),\ O(k^{-1}d^{-3}) \tag{3.5}$$

respectively. It will be assumed that $d>>1$ and kd is of order unity or larger. The last condition is identical to the wave length being of the order d or smaller. (3.2) then reduces to

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{\gamma(\xi)}{x-\xi}\,d\xi = -ke^{-kd}\cos(kx+\alpha) \tag{3.6}$$

The general solution of this singular integral equation is well known (see for example Newman, 1977, p.184). When (3.4) is fulfilled, the solution of (3.6) is

$$\gamma(x) = \frac{2}{\pi}\,\frac{1}{(1-x^2)^{\frac{1}{2}}}\int_{-1}^{1}\frac{ke^{-kd}\cos(k\xi+\alpha)(1-\xi^2)^{\frac{1}{2}}}{x-\xi}\,d\xi \tag{3.7}$$

To find the motion for $x \to -\infty$, we apply (2.2). For the dimensionless $f_1(z)$ we then obtain

$$f_1(z) = \frac{4ke^{-2kd}}{\pi}\left[e^{-ikz} \int_{-1}^{1} \frac{d\eta\, e^{ik\eta}}{(1-\eta^2)^{\frac{1}{2}}} \fint_{-1}^{1} \frac{\cos(k\xi+\alpha)(1-\xi^2)^{\frac{1}{2}}}{\eta-\xi} d\xi\right] \quad x \to -\infty \qquad (3.8)$$

To evaluate (3.8), we change the order of integration. It may be shown that

$$\fint_{-1}^{1} \frac{e^{ik\eta} d\eta}{(\eta-\xi)(1-\eta^2)^{\frac{1}{2}}} = i\pi e^{ik\xi} \int_0^k e^{-i\kappa\xi} J_0(\kappa) d\kappa \qquad (3.9)$$

where J_0 is the Bessel function of order zero. Furthermore we utilize the formula

$$\int_{-1}^{1} e^{iu\xi}(1-\xi^2)^{\frac{1}{2}} d\xi = \pi \frac{J_1(u)}{u},$$

where J_1 denotes the Bessel function of order one. We then get for $f_1(z)$

$$f_1(z) = e^{-ikz}(iBe^{-ik} + iB_2 e^{i\alpha}) \quad x \to -\infty \qquad (3.10)$$

where

$$B_1 = 2\pi k e^{-2kd} \int_0^k \frac{J_0(\kappa)J_1(\kappa)}{\kappa} d\kappa,$$

$$B_2 = 2\pi k e^{-2kd} \int_0^k \frac{J_0(\kappa)J_1(2k-\kappa)d\kappa}{2k-\kappa} \qquad (3.11)$$

The integrals may after some manipulation be reduced to

$$\int_0^k \frac{J_0(\kappa)J_1(\kappa)d\kappa}{\kappa} = k(J_0^2(k)+J_1^2(k)) - J_0(k)J_1(k) \qquad (3.12)$$

$$\int_0^k \frac{J_0(\kappa)J_1(2k-\kappa)}{2k-\kappa} d\kappa = J_0(k)J_1(k) \qquad (3.13)$$

A series expansion gives

$$B_1 = \pi k^2 e^{-2kd}[1 - \frac{1}{8}k^2 + \frac{1}{96}k^4 - \frac{5}{96^2}k^6 + \ldots]$$
$$B_2 = \pi k^2 e^{-2kd}[1 - \frac{3}{8}k^2 + \frac{5}{96}k^4 - \frac{35}{96^2}k^6 + \ldots] \qquad (3.14)$$

The total non-dimensional velocity potential, $f(z)$, is obtained by adding to $f_1(z)$ the contribution corresponding to (3.1). We have

$$f(z) = e^{-ikz}[(1+iB_1)e^{-i\alpha}+iB_2e^{i\alpha}] \qquad x \to -\infty \qquad (3.15)$$

As in the previous sections we add two standing waves to obtain an outgoing wave. The real velocity potential, ϕ, for the outgoing wave satisfies

$$\phi = e^{ky}\cos(kx+\alpha)\cos\sigma t + e^{ky}\sin(kx+\alpha)\sin\sigma t$$
$$= e^{ky}\cos(kx-\sigma t+\alpha) \qquad x \to \infty \qquad (3.16)$$

At $x \to -\infty$ we obtain from (3.15)

$$\phi = \cos\sigma t\ \mathrm{Re}f(z) + \sin\sigma t\ \mathrm{Re}g(z) \qquad x \to -\infty \qquad (3.17)$$

where

$$g(z) = e^{-ikz}[i(1+iB_1)e^{-i\alpha} + B_2e^{i\alpha}] \qquad (3.18)$$

Hence,

$$\phi = \sqrt{1+B_1^2}\ e^{ky}\cos(kx-\sigma t+\alpha-\beta_1) +$$
$$B_2e^{ky}\cos(kx+\sigma t-\alpha-\beta_2) \qquad x \to -\infty \qquad (3.19)$$

where

$$\mathrm{tg}\beta_1 = B_1 \qquad B_2 = \frac{\pi}{2}$$

From (3.19) we may formally write down the coefficient of reflection, R and the coefficient of transmission, T. It is, however, important to keep in mind that (3.19) is an approximate result. The first approximation, $1+iB$ in (3.15), will in a higher approximation most likely be replaced by a term $1+\varepsilon+iB_1$ where

ε is real and $|\varepsilon|$ less than $|B_1|$. The expression $\sqrt{1+B_1^2}$ is then replaced by $\sqrt{1+B_1^2+2\varepsilon}$. Even if $|\varepsilon|$ is as small as $|B_1^2|$ will this term be important in (3.19). The conclusion is that the formula found for T is not correct even in the first approximation. The value for R is, however, correct to the first order, i.e. $R=|B_2|$.

Energy considerations give the following relation

$$R^2 + T^2 = 1 \tag{3.20}$$

In the first approximation we then have

$$R = |B_2|, \quad T = \sqrt{1-B_2^2} \tag{3.21}$$

where B_2 is given by (3.11) and (3.13), or the series (3.14).

From the momentum equation the time average horizontal second order force is given by

$$F = \frac{1}{2}\rho g A^2 R^2 \tag{3.22}$$

Hence, from (3.21)

$$\frac{F}{\rho g A^2} = \frac{1}{2}B_2^2 = 2\pi^2 k^2 e^{-4kd} J_0^2(k) J_1^2(k) \tag{3.23}$$

We see that F is zero for an infinite number of k-values, corresponding to the zeroes of $J_0(k)$ and $J_1(k)$.

We note from (3.7) that $\gamma(x)$ is infinite at $|x|=1$. Hence also the tangential velocity, and the pressure is infinite at these points. The infinite large pressure is acting on an infinite small area, such that F becomes finite. The question naturally arises whether the solution above may be obtained by a limiting process, for example from an ellipse with vanishing minor axis. The case of an ellipse has been discussed by Leppington and Siew (1980) for exactly the same range of parameters as in the present case. Their method, being quite different from the method applied here, does not, however, lead to a definite answer for the reflection coefficient and the force F, so they have to conjecture the form of R. We shall therefore in the next section also give a short account of the application of the theory to an elliptic contour.

4. The elliptic contour

For simplicity we assume that the major axis is parallel to the free surface. As in the previous section we apply non-dimensional quantities. As characteristic length scale we choose half the length of the major axis. Let b be dimensionless and denote half the minor axis. Correspondingly, d is the dimensionless distance between the free surface and the major axis. As in the previous section we assume that $d \gg 1$ and kd is of order unity or larger. The integral equation (1.22) then takes the form

$$\gamma(s') + \mathrm{Im}\frac{1}{\pi}\int_B \frac{\gamma(s)e^{i\beta(s')}}{\zeta(s')-\zeta(s)}\,ds = -\mathrm{Im}2ke^{-i(k\zeta(s')-\beta(s')+\alpha)} \tag{4.1}$$

The equation for the ellipse may be written on parameter form as

$$\zeta(\theta) = -id + ib\sin\theta + \cos\theta \tag{4.2}$$

Introducing this in (4.1), we obtain

$$-\kappa(\theta') + \frac{b}{2\pi}\int_0^{2\pi}\frac{\kappa(\theta)}{1-\lambda^2\cos^2\frac{\theta'+\theta}{2}}\,d\theta = f(\theta') \tag{4.3}$$

where

$$\kappa(\theta) = \gamma(\theta)(\sin^2\theta+b^2\cos^2\theta)^{\frac{1}{2}} \tag{4.4}$$

$$\lambda^2 = 1 - b^2 \tag{4.5}$$

and

$$f(\theta) = -2k(\sin^2\theta+b^2\cos^2\theta)^{\frac{1}{2}}\mathrm{Im}e^{-i(k\zeta(\theta)-\beta(\theta)+\alpha)} \tag{4.6}$$

It is noted that (4.3) is a symmetrical integral equation. Since symmetrical integral equation has at least one real eigenvalue, we could expect that the solution of (4.3) does not exist for all values of b. This is, however, not the case. It may be shown that since b is positive, (4.3) has always a unique solution.

We expand $\kappa(\theta)$ in a Fourier series

$$\kappa(\theta) = \sum_{n=1}^{\infty} (C_n e^{in\theta} + \bar{C}_n e^{-in\theta}) \tag{4.7}$$

The kernel in (4.3) may be expanded in a convergent power series with respect to $\lambda^2 \exp(\pm i(\theta'+\theta))$. Introducing this series and (4.7) in (4.3) we find

$$-\kappa(\theta') + \sum_{n=1}^{\infty} \left(\frac{1-b}{1+b}\right)^{2n} (C_n e^{-in\theta'} + \bar{C}_n e^{in\theta}) = f(\theta') \tag{4.8}$$

By a Fourier transformation of (4.8)

$$-C_m + \bar{C}_m \left(\frac{1-b}{1+b}\right)^m = f_m \qquad m=1,2,\ldots \tag{4.9}$$

where

$$f_m = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-im\theta} d\theta \tag{4.10}$$

Hence

$$C_m = \left[\left(\frac{1+b}{1-b}\right)^m f_m + \bar{f}_m\right] \frac{(1-b^2)^m}{(1-b)^{2m} - (1+b)^{2m}} \tag{4.11}$$

Introducing (4.6) for $f(\theta)$, (4.10) takes the form

$$f_m = \frac{k}{2\pi} e^{-kd} [e^{-i\alpha} H_m + e^{i\alpha} \tilde{H}_m] \tag{4.12}$$

where

$$H_m = -\frac{im}{k} \int_0^{2\pi} e^{-\frac{ik}{2}(1+b)e^{i\theta} - \frac{ik}{2}(1-b)e^{-i\theta} - im\theta} d\theta \tag{4.13}$$

$$\tilde{H}_m = -\frac{im}{k} \int_0^{2\pi} e^{\frac{ik}{2}(1-b)e^{i\theta} + \frac{ik}{2}(1+b)e^{-i\theta} - im\theta} d\theta \tag{4.14}$$

According to (2.2), the motion at $x \to -\infty$ is determined by the integral

$$\int_B \gamma(s) e^{ik\bar{\zeta}(s)} ds = e^{-kd} \int_0^{2\pi} \kappa(\theta) e^{kb \sin\theta + ik\cos\theta} d\theta \tag{4.15}$$

Introducing (4.7) for $\kappa(\theta)$, changing the order of summation and integration and applying (4.13) and (4.14) we obtain

$$\int_B \gamma((s) e^{ik\bar{\zeta}(s)} ds = -ie^{-kd} k \sum_{m=1}^{\infty} \frac{1}{m}(C_m \bar{H}_m - \bar{C}_m \tilde{H}_m) \tag{4.16}$$

C_m is given by (4.11). It is seen that the right hand side of (4.16) contains products of the type $H_m \cdot \tilde{H}_m$. From (4.13) and (4.14) it is noted that H_m and $\tilde{H}_m$ are closely related to Bessel functions. It is indeed easily derived that

$$H_m = -\frac{2\pi i m}{k}(-i)^m \left(\frac{1+b}{1-b}\right)^{\frac{m}{2}} J_m(k\sqrt{1-b^2}) \tag{4.17}$$

$$\tilde{H}_m = -\frac{2\pi i m}{k}(i)^m \left(\frac{1-b}{1+b}\right)^{\frac{m}{2}} J_m(k\sqrt{1-b^2}) \tag{4.18}$$

where J_m denotes Bessel function of order m. Hence

$$C_m \bar{H}_m = -\frac{2\pi}{k} e^{-kd} m^2 \left(\frac{1+b}{1-b}\right)^m J_m^2(k\sqrt{1-b^2}) e^{-i\alpha} \tag{4.19}$$

$$\bar{C}_m \tilde{H}_m = \frac{2\pi}{k} e^{-kd} m^2 (-1)^{m+1} J_m^2(k\sqrt{1-b^2}) e^{i\alpha} \tag{4.20}$$

By (4.16), (4.19) and (4.20) we obtain

$$f_1(z) = e^{-ikz}(iB_1 e^{-i\alpha} + iB_2 e^{i\alpha})$$

where

$$B_1 = 4\pi e^{-2kd} \sum_{m=1}^{\infty} m \left(\frac{1+b}{1-b}\right)^m J_m^2(k\sqrt{1-b^2}) \tag{4.21}$$

$$B_2 = 4\pi e^{-2kd} \sum_{m=1}^{\infty} m(-1)^{m+1} J_m^2(k\sqrt{1-b^2}) \tag{4.22}$$

Utilizing the relation

$$mJ_m(x) = \tfrac{1}{2}x(J_{m+1}(x)+J_{m-1}(x))$$

the infinite sum for B_2 reduces to

$$B_2 = 2\pi e^{-2kd} k\sqrt{1-b^2}\; J_0(k\sqrt{1-b^2})J_1(k\sqrt{1-b^2}) \tag{4.23}$$

The total complex potential $f(z)$ is, according to (1.15)

$$f(z) = e^{-ikz}(e^{-i\alpha}(1+iB_1) + iB_2e^{i\alpha}))$$

Analogous to the previous section, we deduce that the coefficient of reflection, R, and the coefficient of transmission, T, are given by

$$R = |B_2|, \quad T = \sqrt{1-B_2^2} \tag{4.24}$$

The value for R derived here has the same form as the one conjectured by Leppington and Siew (1980).

As argued in the previous section the time average horizontal second order force is given by (see 3.22)

$$F = \tfrac{1}{2}\rho g A^2 R^2$$

Hence, from (4.23) and (4.24)

$$\frac{F}{\rho g A^2} = 2\pi^2 k^2(1-b^2)e^{-4kd} J_0^2(k\sqrt{1-b^2})J_1^2(k\sqrt{1-b^2}) \tag{4.25}$$

As noted in the case of the flat plate we see that F is zero for an infinite number of k values, corresponding to the zeros of $J_0(k\sqrt{1-b^2})$ and $J_1(k\sqrt{1-b^2})$.

For $b \to 0$ we find that

$$\begin{aligned} B_1 &= 4\pi e^{-2kd} \sum_{m=1}^{\infty} mJ_m^2(k) \\ B_2 &= 2\pi e^{-2kd} J_0(k)J_1(k) \end{aligned} \tag{4.26}$$

We have that

$$\sum_{m=1}^{\infty} mJ_m^2(k) = \frac{1}{2}\int_0^k t\left[J_0^2(t)+J_1^2(t)\right]dt = \frac{k}{2}\left[kJ_0^2(k)+kJ_1^2(k)-J_0(k)J_1(k)\right]$$

Hence

$$B_1 = 2\pi k e^{-2kd}\left[kJ_0^2(k)+kJ_1^2(k)-J_0(k)J_1(k)\right] \qquad (4.27)$$

We note that B_1 and B_2 given by (4.26) and (4.27) are identical to those given by (3.11), (3.12) and (3.13). Hence the solution for the elliptic contour for $b \to 0$ is identical to the solution for the flat plate.

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APPENDIX

$$\operatorname{Im} \int_B \frac{\gamma(s)e^{i\beta(s')}}{\zeta(s)-\zeta(s')} ds = \operatorname{Im} \int_B \frac{\gamma(s)}{\zeta(s)-\zeta(s')} \frac{d\zeta(s')}{ds'} ds$$

We have

$$\operatorname{Im} \frac{d\zeta(s')}{ds'} \frac{1}{\zeta(s)-\zeta(s')} = -\operatorname{Im} \frac{d}{ds'} \ln(\zeta(s')-\zeta(s))$$

We write

$$\zeta(s') - \zeta(s) = re^{i\theta}$$

Thereby

$$\operatorname{Im} \frac{d\zeta(s')}{ds'} \frac{1}{\zeta(s)-\zeta(s')} = -\frac{\partial\theta}{\partial s'} = -\frac{\partial \ln r}{\partial n'} = -\frac{1}{r}\frac{\partial r}{\partial n'}$$

where $\partial/\partial n'$ denotes the normal derivative, positive out from the contour. Choosing the direction of the radius vector r from $\zeta(s)$ to $\zeta(s')$, we have that $\partial r/\partial n' = \cos(n',r)$.

$$\operatorname{Im} \int_B \frac{\gamma(s)e^{i\beta(s')}}{\zeta(s)-\zeta(s')} ds = \int_B \frac{\gamma(s)\cos(n',r)}{r} ds.$$